ON SOME CHARACTERISTICS OF SYMBOLIC LOGIC.

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It is now thirty-five years since the publication of Boole's *Laws of Thought.* It has frequently happened, of course, that a great advance in the work of extending scientific ways of thinking over regions which are still in the hands of the natural man has failed to meet with the recognition that it deserved, but, none the less, every fresh instance of a misfortune of this kind is a fresh subject for regret.

The task which Boole accomplished was the complete solution of the problem:—given any number of statements, involving any number of terms, mixed up indiscriminately in the subjects and the predicates, to eliminate certain of those terms, that is, to see exactly what the statements amount to irrespective of them, and then to manipulate the remaining statements so that they shall read as a description of a certain other chosen term (or terms) standing by itself in a subject or a predicate. Ordinary syllogism consists of elimination in the simplest possible case; when we infer from "all kings are tyrants," and "all tyrants are assassinated," that "all kings are assassinated," what we do is to gather up all the information which is conveyed by both premises together, exclusive of that which concerns tyrants. It is plain that ordinary syllogism is of very little avail when there are several premises of such complexity as this, for instance:—"Every a is one only of the two, x and y, unless it is z or w; in the former of which cases it is both x and y, and in the latter case neither of them.

However great a distaste one may have for Symbolic Logic, and however certain one may be that such complicated states of things as the above sentence describes never arise in nature, it is impossible not to admit that it is a great gift to the powers of the common mind to have devised a method by which one can sit down before half a dozen premises of this kind and know just how to go to work to pick out with absolute certainty everything that is said about anything, with or without allusion to anything else. The unaided mind, with a good deal of struggling and floundering, can do pretty hard work of this kind, if it has unusual powers of concentration, but it cannot do it without extreme effort. An organized method for doing thinking like this (which is simply what Symbolic Logic is) amounts, to say the least, to an immense economy of intellectual work. This problem of Logic was completely solved by Boole.

The underlying process of thought involved in his rule for elimination consists, as might have been predicted, in nothing more nor less than the syllogism; the mental act by which, when a term is so given as to form a safe connecting link between other two terms, that term is dropped and the implied connection between the other two terms is set forth in a single statement, is syllogism and nothing else. The only difficulty which the natural man need feel in cases of any complexity whatever, is to extricate the term to be eliminated from the mass of terms attached to it either conjunctively or disjunctively, and to set it forth by itself in the subject or in the predicate; if he could accomplish that, an ordinary syllogism would suffice to put it to flight. It is this extricating which ordinary logic refuses to concern itself with, and it is this limitation of its scope which is largely the cause of the unreality

and of the remoteness from actual thinking that ordinary logic has in the minds of those who are forced to study it. In everyday life, we have constant occasion for seeing (and we have no difficulty in seeing) the exact equivalence of two such statements as these:

"All students of chemistry are also students of either biology or physics."

"Students of chemistry who do not study physics all study biology."

And yet this is an inference which ordinary Logic takes absolutely no account of. What it amounts to is the changing of a positive disjunctive term in the predicate into a negative conjunctive term in the subject. It is a very simple step for the human mind to take, and yet it is the starting point of the immense command over intricate reasoning which is furnished by the modern developments of Deductive Logic. Common logic itself would gain immensely in scope and freedom, and hence in apparent naturalness, if this process were added to the subjects which it now discusses.

The secret of the great command which Symbolic Logic has over complicated trains of reasoning is wholly contained in the fruitful idea that subject and predicate are not necessarily indivisible wholes, but that they can be broken up and their separate elements shifted at pleasure from one side of the copula to the other. If it were doubted before, by the rigid adherents of Boole, that this is the real essence of the matter, and that the symbolism is merely a convenient, not an essential, tool, it can no longer be doubted since Mr. Keynes (*Studies and Exercises in Formal Logic*) has taken the trouble to write out the whole subject with almost no symbolism at all,—with nothing, in fact, except the absence of a symbol to denote *and* (that is, he writes

a b for *a and b*), and the use of large and small letters to denote positive and negative terms respectively. Any considerable amount of symbolism is therefore not a necessity to the new logic, but it is, all the same, of an immense advantage to it.

But with the introduction of symbols, there suddenly arises a strong feeling of abhorrence in the mind of the regular logician. They change Logic into a branch of mathematics, he says,—and with the suppressed premise that mathematics is something not to be endured. But the charge is a wholly unfounded one. Whether a thing is mathematics or not, depends upon its subject-matter, not upon the accidents of its dress. Every form of deductive reasoning is mathematical in the sense of being highly abstract, and of being subject to formal rules of procedure, which make it possible to carry it on without knowledge of what the things are that one is reasoning about, but it is non-mathematical in the sense that it does not deal with any kind of quantity. It is merely a question of convenience of nomenclature whether one defines mathematics to be that branch of logic which deals with measurable quantities, or logic to be that branch of mathematics which deals only with objects in masses and their qualities, without regard to their number or their size, or whether (what is doubtless preferable) one considers them to be parallel branches of science in general. In any case, the greater or less extent to which the invariable rules by which one *must* reason are shadowed forth by formal rules for the combination of counters which stand for the products of thought, has nothing to do with the question. It happens that mathematics is a subject of such tremendous complexity that it is absolutely necessary for it to have recourse to all sorts of signs and symbols and abbreviations, in order to make it at all

possible for the mind to grapple with it. It does not follow that every other science which finds its advantage in similar means is by that reason turned into mathematics. Modern chemistry makes use of a far more highly developed symbolism than any logician has ever thought of proposing. If the symbolical representation of chemical compounds had happened to receive a distinct name borrowed from mathematics, if it had been called chemical quaternions, for instance, there is no doubt that it would have awakened much repugnance in the minds of conservative chemists, and that it would have had a long struggle to get itself accepted.

What is a symbolic treatment of a subject? It is nothing but a system of abbreviations, having more or less of a pictorial quality, having more or less the character of *icons*, as Mr. Pierce would say, according as that character may be desirable or attainable. Even language is constantly simplifying itself by making use of abbreviated marks for complicated concepts; we said at one time *a moving mass of men;* then we took a single Latin word, *mobile,* for that complex idea; then we abbreviated that word into *mob,* and at present the word mob is, as far as our ordinary consciousness is concerned, nothing but an arbitrary mark for the longer phrase. The great aim which language and science both have always before them, is to enable us to do the greatest possible amount of thinking with the least amount of trouble consistent with perfect clearness. The first aid which the overburdened mind seized hold upon, when the premises thrust upon it were too involved for its easy grasp, was paper and pencil; and it is quite conceivable that some early purists objected to this substitution of a mechanical device for the noble work of reason. It is certain that the early introduction of the signs and symbols of alge-

bra met with very great opposition. It was long before mathematicians were able to reconcile themselves to writing y^5 for $y\ y\ y\ y\ y$; and it was still longer before they ceased writing $y\ y$ instead of y^2; they said, doubtless, that in the case of the second power, y^2 was just as long in writing as $y\ y$, and that the latter was easier of comprehension. But this was at a period when the human mind was not accustomed to bold devices for facilitating its hold over nature. At present, no true lover of science hesitates to avail himself of any well devised scheme for enabling him (with paper and pencil before him) to condense into a single field of vision as much information as possible. And it is very little creditable to the logicians that they have shown an unreasoning horror of a given macroscopic arrangement, merely because it has shown itself to be serviceable in the hands of the mathematicians.

The logicians, it is true, have had some excuse in the fact that the scheme of symbolical reasoning proposed by Boole was immensely more complicated and involved than there was any occasion for. Great as his contribution to science was, Boole had an exaggerated idea of its mysterious inward significance. If he had made less of it, others would without doubt have made more of it. The complete solution of the problem is in reality extremely simple, both in theory and in mechanical execution. The introduction of "functions" and "developments" and the whole idea of inverse processes, while it formed a natural way of looking at the question for the trained mathematician, was wholly unnecessary, and was well calculated to frighten away the mere logician. The whole difficult procedure of Boole has been rendered superfluous by the simpler systems of later writers.

This is not the opinion of Mr. Venn,— the most

voluminous of recent writers on the subject. He says (*Symbolic Logic*, p. xxviii): "Other writers have spoken of their 'systems' and contrasted them with that of Boole; but at present there is, I think, only one thing before the world which can without abuse of language be called a system (unless the single methodical alteration of marking alternatives on the non-exclusive plan be allowed to rank as such), and this is exclusively due to Boole."

In order to make it plain that this is a mistaken view, it is necessary to inquire a little more minutely into what it is that constitutes a system of logic. The inventor of a system of Logic has three distinct tasks to perform:

1. He must take the multitudinous propositions which are handed in to him by common language and reduce them all to a limited number of forms of expression.

2. He must lay down the rules in accordance with which several of these statements are to be united into one, or one is to be broken up into several.

3. He must lay down rules in accordance with which information in regard to some of the terms is dropped while absolutely all the information which does not regard those terms is retained.

These three processes may be called, respectively, expression, combination and elimination. (If any peculiar and unnatural form of expression is to be given to the conclusion—as is the case in Boole's system—that would constitute a separate process.) Expression will consist of two parts: a form must be chosen for putting together the separate terms which go to make up the subject or the predicate, and a form must be chosen for attaching the subject and predicate to each other. Terms are put together in actual thinking in two totally different

ways. I mean two different things according as I say:

"All of my friends are *either* learned *or* virtuous." (1)

Or, "All of my friends are *both* learned *and* virtuous." (2)

It is therefore absolutely necessary to have two different signs for connecting the letters, which are to serve us as symbols for terms. These two signs might be anything whatever; the two that are usually employed are, respectively, *the sign of* $+$, and *the absence of any sign.* These are the signs which the mathematician has appropriated to addition and multiplication of quantities, but their use in logic has only a fictitious connection with their use in mathematics. A very large amount of very useless discussion might have been saved if non-mathematical signs had been employed for logic from the start. It is true that in logic a factor is distributed over a sum, as it is in mathematics,

$$a\,(b + c) = a\,b + a\,c,$$

(it is natural to borrow the names *sum, factor, product,* having borrowed the signs); but so also is a summand distributive over a product,

$$a + b\,c = (a + b)\,(a + c),$$

which is not the case in mathematics. The suggestiveness of the use of the mathematical signs is fully equalled by its deceptiveness.

Nor would the slightest difficulty in the working of the logical algebra be occasioned if the meanings of the signs were interchanged, and if we wrote $a\,b$ for what is either a or b, and $a + b$ for what is both. But the fact remains that there is hardly anything which is easier to write than $+$, and nothing which is easier to write than nothing; and also that to write $a\,b$ for what is both a and b, while it has absolutely no connection with multiplication, does closely

simulate the device of grammar by which we say *green apples* to indicate those things which have both the qualities of apples and the quality of being green. But whatever two pictures might be chosen to stand for these two kinds of term-attachments, *the rules for their manipulation must always be exactly the same.* Two statements which are identical in meaning, respectively, with the two statements given above are:

"Whoever is *both* unlearned *and* non-virtuous is not my friend." (1)

"Whoever is *either* unlearned *or* non-virtuous is not my friend." (2)

Thus, a conjunctive combination in the predicate of the affirmative proposition becomes disjunctive (and of opposite quality) in the subject, and conversely. In other words, the negative of a sum is a product of negatives, and the negative of a product is a sum of negatives; or, expressed symbolically, $\overline{a + b} = \bar{a}\,\bar{b}$, and $\overline{a\ b} = \bar{a} + \bar{b}$; that is, what is not either a or b is both non-a and non-b, and what is not both a and b is either non-a or non-b. This rule is an immediate consequence of the two properties of the negative (namely, that x and $\bar{x}$ are mutually exclusive and that they together exhaust the universe); and it must appear in exactly this form in every system of logic (regard not being had to the unimportant modifications produced by marking alternatives on the exclusive plan).

Now if a system of logic consisted in the treatment of the aggregation and determination of terms alone, it would be correct to say that Boole's system is the only one that exists; but in that case Boole's system could not be credited to Boole. Mr. Venn has himself pointed out that Lambert (*Logische Abhandlungen,* 1781) described these logical relations by these words and represented them by the signs $+$ and $\times$.

As matter of fact, however, this is only the beginning of a system of logic. A proposition has not been stated until the connection between its subject and predicate has been laid down, and it happens that this connection differs from that of aggregation or determination in the fact that it is *not* uniform. Ordinary language supplies us with a countless number of forms of propositions, and ordinary reason picks its way from one to another of them by instinct. Upon examination, it is found that these multiform propositions consist of eight essentially distinct types. Common logic has long since forced all propositions into the fixed mould,— subject, copula, predicate; and of the eight propositions which De Morgan has shown to be necessary to complete expression, it has admitted four. The four which it has not discussed may be got from those which it does recognize by the admission of negative subjects; and since De Morgan's great contribution to logic of the idea of a limited universe, negative subjects ought not to seem to the logicians to be too difficult for discussion. In real life, of course, they are as natural as possible,—"Who will not work, must starve," "What is not cheap is not always good,"—and we have no more difficulty in making real syllogisms out of them than out of the types that are recognized.[1]

Another way of seeing the necessity for their admission is this: every proposition (whatever other import it may have) asserts the existence or the non-existence of a certain combination of qualities;

[1] One of the two propositions with negative subjects can be discussed by the unreformed logician in its contraposed form,— "Those who could not swim were not saved" is the same thing as, "All who were saved could swim,"— but the other, "Whoever is not virtuous is miserable," has for its contraposed form "Whoever is not miserable is virtuous," and the subject is as badly negative as before. The ordinary logic has absolutely incapacitated itself from admitting either of these last two sentences into its scheme.

" all a is b " says that there is no such thing as the combination $a\ \bar{b}$, and "some a is not b " asserts that objects do exist which have the qualities a and $\bar{b}$. But all the combinations of a and b (with their negatives) are four in number,—$a\ b$, $\bar{a}\ \bar{b}$, $\bar{a}\ b$, and $a\ \bar{b}$,—and when the signification of the letters is not known, there is no reason for supposing that a certain two of them have any sort of superiority over the other two. It will therefore take eight propositions to affirm and to deny their existence. But by choosing a proper form of words (that is, by throwing the expression of the relation between the terms virtually into the copula), all these eight different things can be said in terms of a and b only,— that is, without the admission of negative terms at all. The following table gives the four different universal statements that can be made in terms of a and b, together with the four denials of them, which are particular propositions.

FOUR DIFFERENT STATEMENTS OF FACT.

UNIVERSAL.

	NON-SYMMETRICAL.		SYMMETRICAL.	
UNIVERSAL.	(A) All of a is b. All of $\bar{b}$ is $\bar{a}$.	(∀) None but a is b. None but $\bar{b}$ is $\bar{a}$.	(E) None of a is b. $(ab)_0$	(Ǝ) All but a is b. $\infty(a+b)$
PARTICULAR.	(a) Not all of a is b. Not all of $\bar{b}$ is $\bar{a}$.	(ɐ) Some besides a is b. Some besides $\bar{b}$ is $\bar{a}$.	(e) Some of a is b.	(ə) Not all but a is b.

PARTICULAR.

I have elsewhere suggested copula signs for these eight propositions, which indicate by their several characters whether the proposition which they symbolize is universal or particular, positive or negative, symmetrical or non-symmetrical; and which are such that a simple rule shows how to

transform a given proposition from one form into any other.[1] It will be noticed that the four propositions of the right-hand half of the table can be read backwards as well as forwards,—"all but a is b" is the same thing as "all but b is a;" but that is not the case in the left-hand half of the table,—"none but a is b" is not the same thing as "none but b is a," but it is the same thing as "none but $\bar{b}$ is $\bar{a}$," or, on changing the places of the terms their quality must be changed also, in order to get an equivalent statement. We shall express this by saying that the copulas of the left-hand half are non-symmetrical and those of the right-hand half are symmetrical. Of the symmetrical copulas, one can be inserted anywhere in a product, and the other can be inserted anywhere in a sum; "none of $a\ b$ is $c\,(b + d)\ e$" is identical with "none of $a\ b\ c\ (b + d)$ is e," and "all but a is $b\ c + d\ e\ f$" is identical with "all but $a + b\ c$ is $d\ e\ f$," with the same thing for the corresponding particular propositions. Moreover, with the symmetrical forms of expression, nothing is changed by putting all the terms into the subject or the predicate as the case may be, thus: "no a is b" is the same thing as "a which is b is nothing," and may be written, upon occasion, $(a\ b)_0$; and "all but a is b" is the same thing as "everything is either a or b," and may be written, upon occasion, $_\infty(a + b)$.

The table already given exhibits the four different forms of expression for four different statements of fact. The four universal (or the four particular) copulas may be made to give expression to one and the same fact by attaching the proper quality to the terms which enter them. Thus, in the following table, all of the universal propositions are equiva-

[1] Some Proposed Reforms in Common Logic. To appear in *Mind*, January, 1890.

lent in meaning with "no a is b," and all of the particular propositions are equivalent in meaning with "some a is b."

FOUR DIFFERENT FORMS OF EXPRESSION FOR ONE AND THE SAME STATEMENT OF FACT.

UNIVERSAL.

NON-SYMMETRICAL.		SYMMETRICAL.	
(A) All of a is $\bar{b}$. All of b is $\bar{a}$.	(Ɐ) None but $\bar{a}$ is b. None but $\bar{b}$ is a.	(E) None of a is b. $(a\ b)_0$	(Ǝ) All but $\bar{a}$ is $\bar{b}$. $\infty(\bar{a} + \bar{b})$
(a) Not all of a is $\bar{b}$. Not all of b is $\bar{a}$.	(ɐ) Some besides $\bar{a}$ is b. Some besides $\bar{b}$ is a.	(e) Some of a is b.	(ə) Not all but $\bar{a}$ is b.

PARTICULAR.

In other words, the fact, "no a is b," can be expressed, at pleasure, in terms of a and b, $\bar{a}$ and $\bar{b}$, a and $\bar{b}$, or $\bar{a}$ and b; and the same is true of the fact, "some a is b."

But the first requirement of a good working scheme is that it should admit no more variety of expression than is absolutely necessary. A single fact, instead of being expressed, at pleasure, in four different ways, as is done in real life, must be expressed in one way only. *Now the whole subsequent working of a scheme of logic will be totally different according as one or the other of these four different modes of expression is chosen as the normal form.*

For one thing, if all propositions are expressed in either of the symmetrical forms, there will be no distinction between subjects and predicates, and both will be treated in accordance with exactly the same rules; but if either of the non-symmetrical forms is chosen, the rules for subjects and predicates will be, throughout, the reverse of each other. Take, for

instance, the rules for under-statement. From "all of $a + b$ is $c\,d$" it can be inferred that "all of $b\,x$ is $c + y$" (where x and y are anything whatever), but not that "all of $a + b + y$ is $c\ d\ x$;" and the same thing holds for the corresponding particular proposition, "not all of a is b," or "some of a is not b." The rule is, that, with this copula (and its negative) an addend can be dropped or a factor can be inserted in the subject, and a factor can be dropped or an addend inserted in the predicate,—a difficult rule to remember. With the other non-symmetrical copula, the rule is exactly the opposite of this. With either of the symmetrical copulas, only one thing has to be borne in mind for the whole complexus of terms. The combination of two universal propositions is got by adding their terms if the 0-copula is used, and by multiplying them together if the ∞-copula is used; to say that there is no $a\,b$ and at the same time there is no $c\,d$ is the same thing as to say that $(a\ b + c\ d)_0$ but to say that everything is either a or b and at the same time everything is either c or d is to say that $_\infty(a + b)(c + d)$. With the non-symmetrical copulas, statements cannot be combined at all (without loss of content) except by virtual transformation into one of the symmetrical copulas, unless they happen to have like predicates or like subjects. The process of elimination presents corresponding differences of treatment with different copulas.

It is plain that the several copulas are as unlike as light and darkness. The symmetrical copulas have the immense advantage, in point of simplicity, that all the terms which they connect play the same role. The second non-symmetrical copula is merely the obverse of the first and does not invite separate discussion. The first, "all a is b," has the important consideration in its favor that the propositions pre-

sented by nature to the logician are most frequently already in that form. This fact lends an apparent naturalness to this form of expression, and if only questions of moderate difficulty are to be attacked, it might be regarded as a determining consideration. The rules for working with this copula ought to be laid down in any treatise on the subject, because they are the generalization of the affirmative syllogism in the first figure, and the first figure has always been the peculiar favorite of the logician. But when the premises are so complicated that a great deal of transposing between subject and predicate has to be done anyway, it is unwise to strain at the slight additional unnaturalness involved in putting all the terms into the subject or into the predicate, in view of the large amount of mental energy that is set free by the fact that one no longer has to bear in mind a different mode of treatment for terms, according to the place in which they occur.

When Mr. Venn said that there is only one system of Logic, he seems, from a foot-note, to have had in mind only Jevons as a pretended maker of another system; and Jevons's work in Symbolic Logic does certainly not amount to a system, but merely to the absence of a system. Since then Mr. Keynes has published his treatment of the non-symmetrical copula, and he states in his preface that he is peculiarly indebted to Mr. Venn—"not merely by reason of his published writings, but also for most valuable suggestions and criticisms given to me while this book was in progress." From this it might, perhaps, have been inferred that Mr. Venn is now aware that the most difficult problems of Symbolic Logic can be solved with other copulas than the negative symmetrical one, were it not that his latest utterance on the subject is this (*Empirical*

Logic, 1889, p. 230): "In fact, groups of really complicated propositions cannot easily be combined, and their net result completely determined, on any other scheme yet worked out." This, it is true, is a much more guarded statement than his former one, and it is also true that complete combination is less easily effected with either of the non-symmetrical copulas than with the others; but the two symmetrical copulas are exactly on an equality as respects facility of complete combination, and the Logic of the affirmative one was completely worked out, six years ago, by Mr. Mitchell. Mr. Keynes also gives (without noticing its importance) all that is really necessary to the Logic of (universal) propositions beginning in "everything is."

Since the character of a system of Logic is absolutely determined by the character of the copula which is chosen to represent its propositions, it follows that there are four essentially different systems of Logic possible,—provided the form chosen for the expression of the particular proposition is the simple denial of the universal, that is, provided one were always to combine two propositions taken from the same column of the Table. But this is not necessary — it may happen that some advantage is gained by putting together a particular of one couplet and a universal chosen from another, and, as matter of fact, it has happened that a very great advantage has been gained by that means. With this degree of freedom, there are sixteen possible systems of Logic, namely, any one of the four universal propositions, in combination with any one of the four particular propositions, may be taken as the standard form of expression. Have all of these sixteen forms of Symbolic Logic been worked out? Virtually, they have,— or, at least, all which could be of any interest,— and it is therefore no longer a possibility

for any one to invent a new system of Logic.

1. The copula chosen by Boole was the negative symmetrical copula, "no a is b," or "there is no a b," which he wrote $a\, b_{,} = 0$. This statement is really of the nature of a proposition, "a which is b *is* non-existent," not of an equation, and it happens that the equational form of expression, while not close to the real nature of the thing to be expressed, had the unfortunate effect of making people think the subject more mathematical than it was, and hence more unpleasant. For a particular proposition, Boole took the immediate denial of the universal,—"some a is b;" but the Logic of the complex particular proposition, and of its combinations (conjunctive and disjunctive) with the universal proposition, he did not work out at all. Boole's further method of procedure is deserving of great credit as a first attempt, but it is cumbrous and (to the non-mathematician) mysterious to the last degree. The final form has been given to the Logic of this copula by Schröder,[1] and his treatment of the subject ought to have completely superseded that of Boole. That it has not done so is doubtless owing to the accident that he has not yet had an English commentator. Mr. Venn makes some allusion to Schröder, but only to his adoption of the non-exclusive plan for logical addition, which is an unessential, though praiseworthy, feature of his method. It is very singular that Mr. Venn does not seem to be aware that the problem of Logic has been completely and admirably worked out by Schröder. His rule for elimination amounts to exactly the same thing as Boole's, from the necessity of the case, and yet Mr. Venn can say, of Boole's formula for elimination, that it "does not seem to be thus capable of introduction [*i. e.* with

[1] Der Operationskreis der Logikkalkuls. Leipzig, 1877.

non-exclusive addition], except under restrictions which almost amount to doing without it altogether." Schröder has accomplished in a few pages, and with admirable simplicity and closeness to real processes of reasoning, what Boole made a very obscure and tortuous journey of. The teacher of Logic who still thinks it necessary to expound the laborious methods of Boole, for any other than historical purposes, is doing his students a serious injury.

I have added the logic of the corresponding particular proposition[1] to Schröder's development of the logic of "no *a* is *b*." The subject is left in a very incomplete state as long as the particular proposition remains undiscussed. Mr. Keynes says (*Formal Logic*, p. 313), "Particular propositions are not in themselves of great value; and they may involve us in troublesome questions of 'existence'." But a universal proposition cannot be simply denied except by means of a particular, and that would be a singular world of argument in which no one was ever permitted to enter a discussion who wished to deny anything which had once been said. Particular propositions do not, in real life, often give us information which is interesting in itself, from a scientific point of view; but they do constantly give us information which is of very great importance in guarding us from a belief in universals which had been supposed to be true. Whoever had been in danger of basing conduct on the premise, "all Catholics are followers of Anti-Christ," would, logically, be very much influenced by the discovery that "some Catholics are saints."

Nor is there anything necessarily troublesome in the question of existence. The import of the particular proposition, "some *a* is *b*," is to affirm the

[1]Studies in Logic, by Members of the Johns Hopkins University. Boston, 1883, pp. 17–71.

existence of the compound ab, and hence of its elements, a and b. Particular propositions, therefore, always affirm the existence of their terms (or of an alternation of their terms) when expressed in this way. The import of the universal proposition is, in every case, to deny the existence of a compound, but not necessarily to affirm the existence or the non-existence of its elements. If, in a particular case, it is known, by extraneous considerations, that the existence of the subject is actually implied, that fact can be stated (and reasoned upon) as a separate proposition,—namely a particular proposition, an affirmation of existence, "there are some a's." All the consequences of these conventions in regard to existence are perfectly easily carried out, and any algebra, therefore, which does not provide for the introduction of particulars,—that is, which does not permit the denial of universals—is thoroughly and unnecessarily incomplete.

The copula "no a is b" might have been combined with any one of the three other particular forms of expression. Its combination with "not everything but a is b," instead of with "some a is b," would give rise to an algebra which would be the exact dualistic of Mr. Mitchell's algebra,—or, it would be exactly like it, if the *meanings* of the symbols for addition and multiplication were interchanged. Its combination with either of the non-symmetrical copulas would lead to much confusion in rules, without any compensating advantage. The syllogisms,

No x is p,
Not only s is x,
Not only p is non-s,

and

No p is non-x,
Not only x is s,
Not only p is s,

will illustrate the rules for elimination with this combination of copulas.

2. The logic of the non-symmetrical affirmative copula, "all *a* is *b*," was first worked out by Mr. Maccoll[1]. Nothing is stranger, in the recent history of Logic in England, than the non-recognition which has befallen the writings of this author. The fact that his contributions appeared in a journal which logicians were not in the habit of referring to (his brief article in *Mind* did not do his method justice), the fact that he was not acquainted with the writings of Boole, and the further accident that he considered it a matter of importance to read "all *a* is *b*," which he wrote $a : b$, in the words, "the statement that a thing is *a* implies the statement that it is *b*,"—all doubtless contributed to making him seem foreign to writers trained in the usual schools of logic. The fact that the nature of the connection between the terms in "all *x* is *y*," and between the propositions in "*a* is *b* is-always-followed-by *c* is *d*," is exactly the same, and must be exhibited in the same formal rules of procedure, has nothing to do with the *words* in which the proposition and the sequence may be expressed. But if it had not been for this accidental misfortune, it seems incredible that English logicians should not have seen that the entire task accomplished by Boole has been accomplished by Maccoll with far greater conciseness, simplicity and elegance; and, what is an interesting point, in terms of that copula which is of by far the most frequent use in daily life.

Mr. Keynes has re-written the entire logic of this copula, (with the unimportant modification that he prefers to use the printed word *or* instead of the sign $+$), *without so much as a single allusion to Mr. Maccoll.* Mr. Venn says (*Symbolic Logic,* page 372, foot-note): "After a careful study [of this

[1]Proc. London Math. Soc., Vols. IX., X., XI., 1877–80.

scheme], aided by a long correspondence with the author, I am unable to find much more in it than the introduction of one more scheme of notation to express certain modifications and simplifications of a part of Boole's system." But Mr. Maccoll has completely solved the problem of logic,—to throw the multiform propositions of real life into a single standard form of expression, to condense the information that interests us by the elimination of certain terms which we do not care for, and to state the information which is left in the form of any terms which we happen to wish to see described. The part of Boole's Logic which Mr. Maccoll does not discuss is the inverse operations; but they are an entirely unessential part of the scheme. If anyone cares to interest himself in them, as a matter of idle curiosity, it should be done in an appendix or a foot-note; even in Boole they work merely to obscure the real process of thought, and the actual reasoning, whether formal or not, goes on its way, entirely oblivious of their existence.

Mr. Maccoll chooses for the expression of his particular propositions the simple denial of his universals, and he writes them, very properly, with the sign of negation attached to the affirmative copula; but he does not discuss their treatment in cases of any complexity. Neither does Mr. Pierce, who has worked out independently the Logic of the same copula—at least, his Fourth Process [1], which is the rule for elimination, concerns only universal propositions, and his Sixth Process is not true of both, unless "taken together" were to mean *taken in combination if they are universal and taken in alternation if they are particular*. The combination of the universal "all non-a is b" with "not only a is

[1]Am. Jour. of Mathematics, Vol III. (1880), p. 39.

non-b" as its denial, would give rise to an algebra in which the rules for elimination between a universal and a particular proposition would be exactly the same as for that between two universals. This universal combined with either of the symmetrical particulars would not be interesting.

3. The negative non-symmetrical copula, "none but a is b," would give rise to an algebra in which everything would be exactly the reverse of the algebra of "all a is b."

4. There remains the algebra of the affirmative symmetrical copula,—"all but a is b," or, "everything is either a or b." This was admirably worked out by Mr. Mitchell, the late Professor of Mathematics in Marietta College, when a fellow in the Johns Hopkins University[1]. His algebra would have been exactly like the algebra of "no a is b," with the exception that everywhere addition and multiplication were interchanged, had he not had, for the first time, the extremely happy idea of combining with his universal proposition a particular taken from a different rubric,—namely, of working out the algebra of the combination,

Everything is a or b,
Something is $\bar{a}$ $\bar{b}$.

By this device, the rules for combination and elimination (where they can be effected at all) *are all identically the same, whether the propositions are universal or particular*. This circumstance gives this algebra a very great advantage over any which seeks to combine a universal proposition with its immediate denial as the particular. It is an advantage which is shared by the combination.

No a is b,
Not all but $\bar{a}$ is $\bar{b}$,

[1]Studies in Logic, pp. 72–106.

and, for simple propositions, this would work about as well as that; but for compound propositions, which we now proceed to consider, it happens that this latter couplet of forms of expression is decidedly unnatural.

The four types of universal compound proposition, when completely expressed, are (if P and Q stand for propositions),

A. If P is true, Q is true,
∀. Only when P is true is Q true.
E. Never when P is true is Q true.
Ǝ. Unless P is true, Q is true,
or, 'The possible' implies that P is true or that Q is true.

A and *E* are the commonest forms of speech for simple propositions, but, by a very curious accident of language, it is not *A* and *E* but *A* and *Ǝ* that are the chosen forms for compound propositions, or sequences, as they may be called, and so great is their preponderance that we have an elliptical form of speech for them,—"if P, Q," and "either P or Q," (that is, "if a is b, c is d" and "either a is b, or c is d.") They are, moreover, strange as it may seem, the only forms of the compound proposition which are ever treated by the logician, and so great is the apparent difficulty of these forms even, to one who has not been trained in Symbolic Logic, that a very recent writer of a text-book actually supposes that

Either A is B or C is D,

and

Either A is B or C is sometimes not D,

are propositions which are denials each of the other.

The actual denials of all four ways of saying one and the same thing are as follows (they are, of course, *particular* sequences):

UNIVERSAL.	PARTICULAR.
Ǝ. Either some must dance or all must sing.	ə. 'T is not necessary that some should dance or else all sing.
A. If none dance, all must sing.	a. If none dance, not all need sing.
∀. Only if some dance may some not sing.	ɐ. Not only when some dance may some not sing.
E. 'T is not permitted that none dance when some do not sing.	e. None may dance *and* some may not sing.

The *words* to be chosen are different according as the sequence is matter of logical or merely material following—in the one case we prefer, for instance, to begin A with *if*, in the other with *when*; but the *connection* between the two propositions is, in both cases, of the same formal kind, and subject to the same symbolical treatment. Of Mr. Mitchell's coup-et of sequences,

> *Ǝ*. Either no *a* is *b* or some *c* is *d*.
> *e*. Sometimes some *a* is *b* and no *c* is *d*,
> or, It may be that
> some *a* is *b* *and* that no *c* is *d*,

it happens that the universal is one of the two most natural, and the particular is the very most natural of them all. That is not, of course, true of his simple propositions; "everything is either not a king else a tyrant" is the least frequently natural of all the forms of expression. It would be interesting to explain, if it could be done, why the relations of terms and the relations of propositions have fallen, in real life, into such different grooves as regards expression.

What I hope to have shown is that two systems of logic are not made the same system by the fact that both are systematic methods of procedure, nor yet by the fact that both express the common part and the aggregate of two terms in the same way; that two systems which have in every respect different rules of procedure, basedpon the fundamental tal differences of their several copulas, are, in any

natural sense of the phrase, different systems; that there are sixteen different forms of symbolic logic possible, the combinations of any one of the four universal propositions with any one of the four particular; that eight of them (the combinations of a symmetrical with a non-symmetrical form) are wholly uninteresting—that is, they occasion confusion without securing naturalness; that, of the other eight, what suggests itself most naturally is the combination of each universal with the particular which immediately denies it, but that the combination with the *other* particular of the same symmetry has the extreme advantage, as a working method, of offering one single set of rules for dealing with particulars and universals; that no system of logic is complete which does not take account of particular propositions; that Mr. Mitchell's system of logic combines naturalness with facility of manipulation (for compound as well as simple propositions) in a higher degree than is possible to any other system; and, in the first place, that the lack of interest which logicians are content to feel in Symbolic Logic is unreasonable, and based upon a mistaken conception of its nature.